Stochastic Processes and their Applications 18 (1984) 165–169
North-Holland

SHORT COMMUNICATION

ON MEAN RECURRENCE TIMES OF STATIONARY ONE-DIMENSIONAL DIFFUSION PROCESSES

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Received 16 December 1982

Let $(X_t)_{t\in\mathbb{R}_+}$ be a diffusion on $\mathbb{R}$ which starts in x and assume that a stationary initial distribution exists with continuous density π. Then

$$\pi(x)=\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\frac{E\tau_{(x-\varepsilon,\,x+\varepsilon)}}{EZ_{x,\varepsilon}},$$

where $\tau_{(x-\varepsilon,\,x+\varepsilon)}$ denotes the first exit time of $(x-\varepsilon, x+\varepsilon)$ and $Z_{x,\varepsilon}$ is the time of first return to x after $\tau_{(x-\varepsilon,\,x+\varepsilon)}$.

AMS 1980 Subject Classification: 60 J 60 (60 G 10)

diffusion processes * recurrence times * stationary processes

To motivate the problem consider an aperiodic positively recurrent Markov chain $X=(X_n)_{n\in\mathbb{N}_0}$ on a countable state space I. For $i\in I$ we denote by $\mathbb{E}_i$ expectation with respect to a start in i. Then the mean of the recurrence time T_i,

$$T_i:=\min\{n\in\mathbb{N}: X_n=i\},$$

and the mass π_i assigned to i by the stationary distribution of the chain are related as follows [2, I.6 and I.7]:

$$\pi_i=\frac{1}{\mathbb{E}_iT_i}. \tag{1}$$

If $(X_t)_{t\in\mathbb{R}_+}$ is a Markov chain in continuous time (positively recurrent, with standard transition probabilities and almost all paths right continuous) an analogous result holds: If τ_i,

$$\tau_i:=\inf\{t\in\mathbb{R}_+: X(t)\neq i\},$$

denotes the time of first exit from a stable state $i \in I$ and Z_i,

$$Z_i := \inf\{t \in \mathbb{R}_+ : X(t) = i, \exists s \in (0, t); X(s) \neq i\},$$

denotes the time of the first return to i, then

$$\pi_i = \frac{\mathbb{E}_i \tau_i}{\mathbb{E}_i Z_i}, \tag{2}$$

where π_i again denotes the probability of i under the stationary distribution [1, Proposition 15.56].

The aim of the present note is to prove a corresponding result for one-dimensional diffusion processes.

Let $X = (X_t)_{t \in \mathbb{R}_+}$ be a diffusion process on the real line $\mathbb{R}$ with characteristic operator U,

$$Uf(x) = \mu(x) f'(x) + \tfrac{1}{2}\sigma^2(x) f''(x),$$

where μ and σ^2 are continuous functions on $\mathbb{R}$ and $\sigma^2(x) > 0$ for all $x \in \mathbb{R}$ (we more or less follow Breiman [1]). We assume that a stationary distribution exists. Under the above assumptions this distribution has a unique continuous density π with respect to Lebesgue measure [1, p. 385]. A formula for $\pi(x)$ analogous to (2) will involve some limiting procedure as is also clear from

$$\inf\{t \in \mathbb{R}_+ : X(t) = x, \exists s \in (0, t): X(s) \neq x\} = 0$$

$\mathbb{P}_x$-almost surely, where $\mathbb{P}_x$ denotes that probability measure under which X starts in x. Let $\tau_{(x-\varepsilon, x+\varepsilon)}$,

$$\tau_{(x-\varepsilon, x+\varepsilon)} := \inf\{t \in \mathbb{R}_+ : X(t) \notin (x-\varepsilon, x+\varepsilon)\},$$

and $Z_{x,\varepsilon}$,

$$Z_{x,\varepsilon} := \inf\{t \in \mathbb{R}_+ : X(t) = x, \exists s \in (0, t): X(s) \notin (x-\varepsilon, x+\varepsilon)\},$$

denote the first exit time from $(x-\varepsilon, x+\varepsilon)$ and the time of the first return to x after leaving $(x-\varepsilon, x+\varepsilon)$ respectively.

Heuristically, if we could lump together $[x + i\varepsilon, x + (i+1)\varepsilon)$ for all $i \in \mathbb{Z}$ to a single state without losing the Markov property then $2\varepsilon\pi(x)$ would roughly agree with π_i and (2) would therefore give

$$2\varepsilon\pi(x) \sim \frac{\mathbb{E}_x \tau_{(x-\varepsilon, x+\varepsilon)}}{\mathbb{E}_x Z_{x,\varepsilon}}. \tag{3}$$

This is in general not possible, but we may expect that (3) holds for small ε in some sense. The following theorem confirms this supposition except for the factor 2.

Theorem. *Under the above assumptions*

$$\pi(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \frac{\mathbb{E}_x \tau_{(x-\varepsilon, x+\varepsilon)}}{\mathbb{E}_x Z_{x,\varepsilon}} \tag{4}$$

holds for every $x \in \mathbb{R}$.

Proof. For any interval $I \subset \mathbb{R}$, bounded or not, we denote by τ_I the time of first exit from I,

$$\tau_I := \inf\{t \in \mathbb{R}_+ : X(t) \notin I\}.$$

The assumption $\sigma^2(x) > 0$ for all $x \in \mathbb{R}$ yields for bounded I

$$\mathbb{P}_x(\tau_I < \infty) = 1 \quad \text{for all } x \in I.$$

If we define $s : \mathbb{R} \to \mathbb{R}$ by

$$s(x) := \exp\left(- \int_0^x \frac{2\mu(t)}{\sigma^2(t)} \, \mathrm{d}t \right),$$

and if S has derivative s, then for all $a,x,b \in \mathbb{R}$ with $a \leq x \leq b$, $a < b$,

$$\mathbb{P}_x(X(\tau_{(a,b)}) = b) = \frac{S(x) - S(a)}{S(b) - S(a)}$$

[1, Theorem 16.27 and Proposition 16.29], so we obtain

$$\lim_{\varepsilon \to 0} \mathbb{P}_x(X(\tau_{(x-\varepsilon,\, x+\varepsilon)}) = x + \varepsilon) = \lim_{\varepsilon \to 0} \mathbb{P}_x(X(\tau_{(x-\varepsilon,\, x+\varepsilon)}) = x - \varepsilon) = \tfrac{1}{2}.$$

Now let m,

$$m(x) := \frac{1}{\sigma^2(x)} \exp\left(\int_0^x \frac{2\mu(t)}{\sigma^2(t)} \, \mathrm{d}t \right),$$

denote a density of the corresponding speed measure M of the process.

Then for every bounded interval $I = [a, b] \subset \mathbb{R}$ we have

$$\mathbb{E}_x \tau_I = 2 \int_a^b \frac{(S(b) - S(x \vee y))(S(x \wedge y) - S(a))}{S(b) - S(a)} m(y) \, \mathrm{d}y \tag{5}$$

[1, p. 365 and p. 379].

We first consider the asymptotic behaviour of $\mathbb{E}_x \tau_{(x-\varepsilon,\, x+\varepsilon)}$ as $\varepsilon \to 0$.

The continuity of s and m allows the application of the dominated convergence theorem leading to

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{x-\varepsilon}^{x} (S(y) - S(x - \varepsilon)) m(y) \, \mathrm{d}y$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_x^{x+\varepsilon} (S(x + \varepsilon) - S(y)) m(y) \, \mathrm{d}y = \tfrac{1}{2} s(x) m(x)$$

after an obvious transformation of the range of integration. Using (5) we now obtain

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \mathbb{E}_x \tau_{(x-\varepsilon,\, x+\varepsilon)} = \lim_{\varepsilon \to 0} \frac{S(x+\varepsilon) - S(x)}{S(x+\varepsilon) - S(x-\varepsilon)} \frac{2}{\varepsilon^2} \int_{x-\varepsilon}^{x} (S(y) - S(x-\varepsilon)) m(y) \, \mathrm{d}y$$

$$+ \lim_{\varepsilon \to 0} \frac{S(x) - S(x-\varepsilon)}{S(x+\varepsilon) - S(x-\varepsilon)} \frac{2}{\varepsilon^2} \int_x^{x+\varepsilon} (S(x+\varepsilon) - S(y)) m(y) \, \mathrm{d}y$$

$$= s(x) m(x).$$

Turning to $\mathbb{E}_x Z_{x,\varepsilon}$ we note that the strong Markov property of X entails

$$\mathbb{E}_x Z_{x,\varepsilon} = \mathbb{E}_x \tau_{(x-\varepsilon,\, x+\varepsilon)} + \mathbb{P}_x(X(\tau_{(x-\varepsilon,\, x+\varepsilon)}) = x+\varepsilon)\mathbb{E}_{x+\varepsilon}\tau_{(x,\infty)}$$
$$+\mathbb{P}_x(X(\tau_{(x-\varepsilon,\, x+\varepsilon)}) = x-\varepsilon)\mathbb{E}_{x-\varepsilon}\tau_{(-\infty,\, x)}.$$

The above computations imply

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}_x\tau_{(x-\varepsilon,\, x+\varepsilon)} = 0.$$

We assumed the existence of a stationary distribution for the process. This firstly implies that M is finite, since the stationary distribution is a constant multiple of the speed measure ([1], p. 385) and secondly, under a π-mixture of $\mathbb{P}_x$, $x\in\mathbb{R}$, the distribution of $X(t)$ does not depend on t. We must therefore have $S(x)\to\pm\infty$ as $x\to\pm\infty$, since otherwise X will leave $\mathbb{R}$ in a finite time with probability 1 because of (5). These arguments show that we may use dominated convergence to obtain

$$\begin{aligned}\mathbb{E}_{x+\varepsilon}\tau_{(x,\infty)} &= \lim_{b\to\infty}\mathbb{E}_{x+\varepsilon}\tau_{(x,b)}\\ &= \lim_{b\to\infty} 2\frac{S(b)-S(x+\varepsilon)}{S(b)-S(a)}\int_x^{x+\varepsilon}(S(y)-S(x))m(y)\,\mathrm{d}y\\ &\quad+\lim_{b\to\infty} 2(S(x+\varepsilon)-S(x))\int_{x+\varepsilon}^{b}\frac{S(b)-S(y)}{S(b)-S(x)}m(y)\,\mathrm{d}y\\ &= 2\int_x^{x+\varepsilon}(S(y)-S(x))m(y)\,\mathrm{d}y\\ &\quad+2(S(x+\varepsilon)-S(x))M((x+\varepsilon,\infty)).\end{aligned}$$

This gives

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}_{x+\varepsilon}\tau_{(x,\infty)} = 2s(x)M((x,\infty)),$$

and similarly

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}_{x-\varepsilon}\tau_{(-\infty,x)} = 2s(x)M((-\infty,x)),$$

so we obtain

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}_x Z_{x,\varepsilon} = s(x)M(\mathbb{R}). \tag{6}$$

On using again the fact that m is a multiple of π the assertion of the theorem follows. □

The functions s and m in the above proof may be replaced by any solution s,m of

$$Uf(x)=\tfrac{1}{2}\frac{1}{m(x)}\left(\frac{1}{s(x)}f'(x)\right)',$$

so in general s and m are only unique up to a constant factor. In the case of processes in natural scale, i.e. if $\mu\equiv 0$, $s\equiv 1$ is a canonical choice for s, let M denote the corresponding (unique) speed measure. In this situation equation (6) of the above proof gives a simple probabilistic interpretation of the total mass of M.

References

[1] L. Breiman, Probability (Addison-Wesley, Reading, 1968).
[2] K.L. Chung, Markov Chains With Stationary Transition Probabilities 2nd ed. (Springer, Berlin, 1967).